

SYMMETRY RESULTS FOR COOPERATIVE ELLIPTIC SYSTEMS VIA LINEARIZATION

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Abstract. In this paper we prove symmetry results for classical solutions of nonlinear cooperative elliptic systems in a ball or in annulus in \mathbb{R}^N , $N \geq 2$. More precisely we prove that solutions having Morse index $j \leq N$ are foliated Schwarz symmetric if the nonlinearity is convex and a full coupling condition is satisfied along the solution.

1. Introduction and statement of the results

We consider a semilinear elliptic system of the type

$$(1.1) \quad \begin{cases} -\Delta U = F(|x|, U) & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$$

where $F = (f_1, \dots, f_m)$ is a function belonging to $C^{1,\alpha}([0, +\infty) \times \mathbb{R}^m; \mathbb{R}^m)$ and Ω is a bounded domain in \mathbb{R}^N , $m, N \geq 2$. Here $U = (u_1, \dots, u_m)$ is a vector valued function in Ω .

It is well known that systems of this type arise in many applications in different fields, we refer to [15], [16] for some of these.

We are interested in studying symmetry properties of classical solutions of (1.1) when Ω is rotationally symmetric i.e. a ball or an annulus centered at the origin in \mathbb{R}^N . If Ω is a ball and the system is cooperative then by using the famous "moving plane method" ([21], [10]) it can be proved (see [4], [6], [7], [23]) that every positive solution U (i.e. $U_i > 0$ in Ω for any $i = 1, \dots, m$) is radial and radially decreasing under the additional hypothesis that each f_i is nonincreasing with respect to $|x|$. Here we consider solutions of any sign, do not require f to be monotone in $|x|$ and would like to obtain results also in the case of an annulus in the same direction of the results obtained in [17] and [19] for scalar equations. We recall that in [17] and [19] (see also [11]) it was proved that classical solutions with Morse index less than or equal to N are foliated Schwarz symmetric if the nonlinearity is convex or its first derivative is convex and Ω is a ball or an annulus.

2010 *Mathematics Subject Classification.* 35B06, 35B50, 35J47, 35G60.

Key words and phrases. Cooperative elliptic systems, Symmetry, Maximum Principle, Morse index.

Supported by PRIN-2009-WRJ3W7 grant.

Another symmetry result obtained by symmetrization for some particular systems, is contained in [13].

Let us give the definition of a foliated Schwarz symmetric function.

DEFINITION 1.1. Let Ω be a rotationally symmetric domain in \mathbb{R}^N , $N \geq 2$. We say that a continuous vector valued function $U = (u_1, \dots, u_m) : \Omega \rightarrow \mathbb{R}^m$ is foliated Schwarz symmetric if each component u_i is foliated Schwarz symmetric with respect to the same vector $p \in \mathbb{R}^N$. In other words there exists a vector $p \in \mathbb{R}^N$, $|p| = 1$, such that $U(x)$ depends only on $r = |x|$ and $\theta = \arccos\left(\frac{x}{|x|} \cdot p\right)$ and U is (componentwise) nonincreasing in θ .

Remark 1.1. Let us observe that if U is a solution of (1.1) and the system satisfies some coupling conditions, as required in Theorem 1.1, then the foliated Schwarz symmetry of U implies that either U is radial or it is strictly decreasing in the angular variable θ . This will be deduced by the proof of Theorem 1.1.

In order to state our results we need to define the Morse index for solutions of (1.1) and the coupling conditions for the system. However they will be restated and commented in Section 2 for general semilinear elliptic systems.

DEFINITION 1.2. i) Let U be a $C^2(\Omega; \mathbb{R}^m)$ solution of (1.1). We say that U is linearized stable (or that it has zero Morse index) if the quadratic form

$$(1.2) \quad \begin{aligned} Q_U(\Psi; \Omega) &= \int_{\Omega} [|\nabla \Psi|^2 - J_F(|x|, U)(\Psi, \Psi)] dx = \\ &\int_{\Omega} \left[\sum_{i=1}^m |\nabla \psi_i|^2 - \sum_{i,j=1}^m \frac{\partial f_i}{\partial u_j}(|x|, U(x)) \psi_i \psi_j \right] dx \geq 0 \end{aligned}$$

for any $\Psi = (\psi_1, \dots, \psi_m) \in C_c^1(\Omega; \mathbb{R}^m)$ where $J_F(x, U)$ is the jacobian matrix of F computed at U .

ii) U has (linearized) Morse index equal to the integer $\mu = \mu(U) \geq 1$ if μ is the maximal dimension of a subspace of $C_c^1(\Omega; \mathbb{R}^m)$ where the quadratic form is negative definite.

DEFINITION 1.3. • We say that the system (1.1) is cooperative or weakly coupled in an open set $\Omega' \subseteq \Omega$ if

$$\frac{\partial f_i}{\partial u_j}(|x|, u_1, \dots, u_m) \geq 0 \quad \forall (x, u_1, \dots, u_m) \in \Omega' \times \mathbb{R}^m$$

for any $i, j = 1, \dots, m$ with $i \neq j$.

• We say that the system (1.1) is fully coupled along a solution U in an open set $\Omega' \subseteq \Omega$ if it is cooperative in Ω' and in addition $\forall I, J \subset \{1, \dots, m\}$ such that $I \neq \emptyset$, $J \neq \emptyset$, $I \cap J = \emptyset$, $I \cup J =$

$\{1, \dots, m\}$ there exist $i_0 \in I, j_0 \in J$ such that

$$\text{meas} \left(\{x \in \Omega' : \frac{\partial f_{i_0}}{\partial u_{j_0}}(|x|, U(x)) > 0\} \right) > 0$$

As it will be clear in the following sections the previous definition reflects the fact that the linearized system at a solution U is weakly or fully coupled, according to Definition 2.1.

Let us point out that the last definition is slightly different from the usual one stated in the literature, because the "full coupling conditions" on the derivatives are required only when computed on the solution but on a subset Ω' . It is not difficult to see that there are systems which are fully coupled along some solution but not fully coupled in the usual sense. This was observed and used in [14] and we will give some examples in Section 4.

Let $e \in S^{N-1}$ be a direction, i.e. $e \in \mathbb{R}^N, |e| = 1$, and let us define the set

$$\Omega(e) = \{x \in \Omega : x \cdot e > 0\}$$

THEOREM 1.1. *Let Ω be a ball or an annulus in \mathbb{R}^N , $N \geq 2$, and let $U \in C^{3,\alpha}(\overline{\Omega}; \mathbb{R}^m)$ be a solution of (1.1) with Morse index $\mu(U) \leq N$. Moreover assume that:*

- i) *The system is fully coupled along U in $\Omega(e)$, for any $e \in S^{N-1}$.*
- ii) *For any $i, j = 1, \dots, m$ $\frac{\partial f_i}{\partial u_j}(|x|, u_1, \dots, u_m)$ is nondecreasing in each variable u_k , $k = 1, \dots, m$, for any $|x| \in \Omega$.*
- iii) *if $m \geq 3$ then, for any $i \in \{1, \dots, m\}$, $f_i(|x|, u_1, \dots, u_m) = \sum_{k \neq i} g_{ik}(|x|, u_i, u_k)$ where $g_{ik} \in C^{1,\alpha}([0, +\infty) \times \mathbb{R}^2)$.*

Then U is foliated Schwarz symmetric and if U is not radial then it is strictly decreasing in the angular variable (see Definition 1.1).

Remark 1.2. • Requiring that the system is fully coupled along U in $\Omega(e)$, for any $e \in S^{N-1}$, is an assumption easily satisfied by most of the systems encountered in applications (see Section 4). It is essentially needed to ensure the validity of the strong maximum principle in the domains $\Omega(e)$.

- The monotonicity hypothesis ii) on the derivatives $\frac{\partial f_i}{\partial u_j}$ in Theorem 1.1 implies that each f_i is convex with respect to each variable u_j , $i, j = 1, \dots, m$.
- The assumption iii), which states that each f_i is the sum of functions which depend only on u_i and one of the other variables u_k , is obviously not needed if $m = 2$.

Hypotheses ii) and iii) together imply the following conditions, that will be exploited in the proof:

- a) $\frac{\partial f_i}{\partial u_i}(|x|, u_1, \dots, u_m)$ ($= \sum_{k \neq i} \frac{\partial g_{ik}}{\partial u_i}(|x|, u_i, u_k)$ if $m \geq 3$) is nondecreasing in u_i for any $i = 1, \dots, m$, i.e. if $U = (u_1, \dots, u_i, \dots, u_m)$ and $\overline{U} = (u_1, \dots, \overline{u}_i, \dots, u_m)$ with $u_i \leq \overline{u}_i$ then $\frac{\partial f_i}{\partial u_i}(|x|, U) \leq \frac{\partial f_i}{\partial u_i}(|x|, \overline{U})$ for any $x \in \Omega$.

- b) If $i \neq j$ $\frac{\partial f_i}{\partial u_j}(|x|, u_1, \dots, u_m)$ ($= \frac{\partial g_{ij}}{\partial u_j}(|x|, u_i, u_j)$ if $m \geq 3$) depends only on u_i, u_j and it is nondecreasing in u_i, u_j , i.e. if $U = (u_1, \dots, u_m)$, $V = (v_1, \dots, v_m)$ and $u_i \leq v_i, u_j \leq v_j$, then $\frac{\partial f_i}{\partial u_j}(|x|, U) \leq \frac{\partial f_i}{\partial u_j}(|x|, V)$ for any $i, j = 1, \dots, m$ for any $x \in \Omega$.

Obviously the previous theorem holds in particular for stable solutions. However in this case, as for scalar equations, it is not difficult to see that the solution is radial, even without the assumption ii) and iii). Therefore we have the following

THEOREM 1.2. *If the system is fully coupled along a stable solution U in Ω , then U is radial.*

We shall see in the proof of Theorem 1.1 that for nonradial Morse index one solutions the following conditions hold.

COROLLARY 1.1. *Under the assumptions of Theorem 1.1 if a solution U has Morse index one and it is not radial then necessarily*

$$(1.3) \quad \sum_{j=1}^m \frac{\partial f_i}{\partial u_j}(r, U(r, \theta)) \frac{\partial u_j}{\partial \theta}(r, \theta) = \sum_{j=1}^m \frac{\partial f_j}{\partial u_i}(r, U(r, \theta)) \frac{\partial u_j}{\partial \theta}(r, \theta)$$

for any $i = 1, \dots, m$, with (r, θ) as in Definition 1.1.

In particular if $m = 2$ then (1.3) implies that

$$(1.4) \quad \frac{\partial f_1}{\partial u_2}(|x|, U(x)) = \frac{\partial f_2}{\partial u_1}(|x|, U(x)), \quad \forall x \in \Omega$$

Remark 1.3. The result of Corollary 1.1 is somewhat surprising because it asserts, under the hypotheses of Theorem 1.1, that for any nonradial Morse index one solution another coupling condition, namely (1.3) (and in particular (1.4) if $m = 2$), must hold along the solution. Some consequences of this, for a particular system, will be illustrated in Section 4.

Let us now explain the strategy of the proof of Theorem 1.1.

As in the scalar case the key idea, introduced in [17], to get the symmetry of the solution, exploiting the information on its Morse index, is to use convexity assumptions on the nonlinearity to relate the linearized operator at the solution with the linear operator that arises when considering the difference between the solution and its reflection with respect to an hyperplane passing through the origin (see [17], [18], [19], [11]). To do this, in the scalar case and for bounded domains Ω , the information on the quadratic form associated to the linearized operator at the solution, deduced by its Morse index, is exploited by means of the eigenvalues of the linearized operator. Indeed this operator is selfadjoint and so there is a variational characterization of the eigenvalues which links them to the quadratic form. This is partially

used also in the unbounded domain case (see [11]) even if the spectrum cannot be easily described, exploiting the eigenvalues of the linearized operator in large balls.

In the case of systems this no longer holds. Indeed if the Jacobian matrix $J_F(|x|, U)$ is not symmetric the linearized operator L_U (see (3.3)) is not selfadjoint, even in bounded domains. Then no variational characterization of eigenvalues can be exploited and hence the scalar case approach cannot be straightforward followed.

To bypass this difficulty we associate to the Jacobian matrix $J_F(|x|, U)$ its symmetric part $\frac{1}{2}(J_F(|x|, U) + J_F^t(|x|, U))$, where J_F^t is the transpose of the matrix J_F (see (2.8) in the linear case), which then gives rise to the selfadjoint operator \tilde{L}_U (see (3.4)) whose spectrum can be variationally characterized.

The crucial, simple remark is that the quadratic form associated to the linearized operator L_U is the same as the quadratic form associated to the selfadjoint operator \tilde{L}_U .

Therefore the eigenvalues of \tilde{L}_U can be exploited to study the symmetry of the solution U , using the information on the Morse index. However, though essential in our proofs, the consideration of these eigenvalues is not sufficient to reach the conclusion of our theorems, but we also need to use the principal eigenvalue of the linearized operator, that can be defined for general linear operators (see [3], [5], [22]). This can be understood by the fact that the positivity of this eigenvalue is a necessary and sufficient condition for the (weak) maximum principle to hold (see Proposition 2.2) and the maximum principle is another key-ingredient in our proofs.

Obviously if the jacobian matrix $J_F(|x|, U)$ is symmetric then the linearized operator is selfadjoint and the classical spectral theory holds, in particular the principal eigenvalue coincides with the first eigenvalue and all proofs are simpler.

This happens, for example, when the system is of *gradient type*, i.e. when $F = \text{grad } g$ for some scalar function g (see [7]). In this case the linearized operator corresponds to the second derivative of a suitable associated functional.

However this is not the case for many interesting systems, like e.g. the so called *hamiltonian systems* (see [7] and the references therein).

Another important remark about the assumptions of our theorems is that, as in previous symmetry results for systems (see [4], [6], [13]), coupling conditions on the system are necessary. Indeed it is easy to construct counterexamples to the symmetry if they do not hold.

In particular, since the definition of foliated Schwarz symmetry for vector-valued functions requires that all components are foliated Schwarz symmetric with respect to the same vector, a *strong* coupling condition is needed.

Finally let us make a general comment on symmetry theorems of the type described in this paper.

As for the scalar case, to apply our symmetry results, some information on the Morse index of the solution is needed. So the question is: how to get it?

If the system is of gradient type then, often, variational methods, used to prove the existence of solutions, also carry information on the Morse index ([1], [7], [14]). A standard example is given by solutions obtained by the Mountain Pass theorem.

If the system is not of this type it could happen that using variational methods, one ends up with the second derivative of a functional which is strongly indefinite. This is for example the case of the so called Hamiltonian systems.

However this does not mean that solutions do not have finite (linearized) Morse index, because the linearized operator is not the second derivative of such functionals. Then the linearized operators and its Morse index could be studied by some other methods, like continuation techniques. We give some examples in Section 4.

Since the study of the linearized system is also important to understand the orbital stability of the solution and informations on the Morse index are useful to get qualitative properties of solutions, we believe that further investigation in this direction should be done.

The outline of the paper is the following. In Section 2 we state and prove some important results on the spectral theory and maximum principles for linear systems. Most of them are either well known or could be easily deduced by known theorems, but we think that is worth to recall and collect them together to make the paper self-contained and to avoid to give vague references to the reader.

In Section 3 we show preliminary results on our semilinear system (1.1) and prove our symmetry results.

In Section 4 we present a few examples and make further comments.

2. Preliminaries on linear systems: spectral theory and maximum principles

Let Ω be any smooth bounded domain in \mathbb{R}^N , $N \geq 2$, and D a $m \times m$ matrix with bounded entries:

$$(2.1) \quad D = (d_{ij})_{i,j=1}^m, \quad d_{ij} \in L^\infty(\Omega)$$

Let us consider the linear elliptic system

$$(2.2) \quad \begin{cases} -\Delta U + D(x)U = F & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$$

i.e.

$$\begin{cases} -\Delta u_1 + d_{11}u_1 + \cdots + d_{1m}u_m = f_1 & \text{in } \Omega \\ \dots & \dots \\ -\Delta u_m + d_{m1}u_1 + \cdots + d_{mm}u_m = f_m & \text{in } \Omega \\ u_1 = \cdots = u_m = 0 & \text{on } \partial\Omega \end{cases}$$

where $F = (f_1, \dots, f_m) \in (L^2(\Omega))^m$, $U = (U_1, \dots, U_m)$.

This kind of linear system appears in the linearization of the semilinear elliptic system (1.1).

DEFINITION 2.1. The system (2.2) is said to be

- *cooperative or weakly coupled* in Ω if

$$(2.3) \quad d_{ij} \leq 0 \text{ a.e. in } \Omega, \quad \text{whenever } i \neq j$$

- *fully coupled* in Ω if it is weakly coupled in Ω and the following condition holds:

$$(2.4) \quad \begin{aligned} &\forall I, J \subset \{1, \dots, m\}, I, J \neq \emptyset, I \cap J = \emptyset, I \cup J = \{1, \dots, m\} \\ &\exists i_0 \in I, j_0 \in J : \text{meas}(\{x \in \Omega : d_{i_0 j_0} < 0\}) > 0 \end{aligned}$$

It is well known that either condition (2.3) or conditions (2.3) and (2.4) together are needed in the proofs of maximum principles for systems (see [6], [8], [22] and the references therein). In particular if both are fulfilled the strong maximum principle holds as it is shown in the next theorem.

Notation remark: here and in the sequel inequalities involving vectors should be understood to hold componentwise, e.g. if $\Psi = (\psi_1, \dots, \psi_m)$, Ψ nonnegative means that $\psi_j \geq 0$ for any index $j = 1, \dots, m$.

THEOREM 2.1. (*Strong Maximum Principle and Hopf's Lemma*). Suppose that (2.1), (2.3) and (2.4) hold and $U = (u_1, \dots, u_m) \in C^1(\bar{\Omega}; \mathbb{R}^m)$ is a weak solution of the inequality

$$-\Delta U + D(x)U \geq 0 \text{ in } \Omega$$

i.e.

$$(2.5) \quad \int_{\Omega} \nabla U \cdot \nabla \Psi + D(x)(U, \Psi) = \int_{\Omega} \left[\sum_{i=1}^m \nabla u_i \cdot \nabla \psi_i + \sum_{i,j=1}^m d_{ij}(x) u_i \psi_j \right] dx \geq 0$$

for any nonnegative $\Psi = (\psi_1, \dots, \psi_m) \in C_c^1(\Omega; \mathbb{R}^m)$.

If $U \geq 0$ in Ω , then either $U \equiv 0$ in Ω or $U > 0$ in Ω . In the latter case if $P \in \partial\Omega$ and $U(P) = 0$ then $\frac{\partial U}{\partial \nu}(P) < 0$, where ν is the unit exterior normal vector at P .

Proof. It suffices to apply the scalar strong maximum principle using (2.3) to obtain that each component u_k is either identically equal to

zero or positive, and then to use (2.4) to prove that the same alternative holds for all the other components. To be more precise, for any equation we have $-\Delta u_j + d_{jj}u_j \geq \sum_{i \neq j} -d_{ij}u_i \geq 0$, since $d_{ij} \leq 0$ if $i \neq j$ and $u_i \geq 0$. This implies that, for any $j = 1, \dots, m$, either $u_j \equiv 0$ or $u_j > 0$ and in the latter case by Hopf's Lemma we have the sign of the normal derivative on a point of the boundary where the function u_j vanishes. Suppose now that U does not vanish identically in Ω and let $J \subset \{1, \dots, m\}$ the set of indexes j such that $u_j > 0$ in Ω . Suppose by contradiction that J is a proper subset of $\{1, \dots, m\}$, and let $I = \{1, \dots, m\} \setminus J$ the set of indexes i such that $u_i \equiv 0$. If i_0, j_0 are as in (2.4) then $-\Delta u_{i_0} + d_{i_0 i_0} u_{i_0} \geq -\sum_{j \neq i_0} -d_{i_0 j} u_j \geq -d_{i_0, j_0} u_{j_0} \neq 0$, so that by the scalar strong maximum principle we get $u_{i_0} > 0$, which is a contradiction. \square

Let us recall that weak maximum principles, Harnack inequalities and other estimates have been studied in many papers with general conditions and also for elliptic operators not in divergence form (see [22] and the references therein).

We are interested in the quadratic form associated with system (2.2), namely

$$(2.6) \quad Q(\Psi; \Omega) = \int_{\Omega} [|\nabla \Psi|^2 + D(x)(\Psi, \Psi)] dx = \int_{\Omega} \left[\sum_{i=1}^m |\nabla \Psi_i|^2 + \sum_{i,j=1}^m d_{ij}(x) \Psi_i \Psi_j \right] dx$$

for $\Psi \in C_c^1(\Omega; \mathbb{R}^m)$ (or $\Psi \in H_0^1(\Omega; \mathbb{R}^m)$).

It is easy to see that this quadratic form coincides with the quadratic form associated to the symmetric system

$$(2.7) \quad \begin{cases} -\Delta U + C(x)U = F & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$$

i.e.

$$\begin{cases} -\Delta u_1 + c_{11}u_1 + \dots + c_{1m}u_m & = f_1 \\ \dots & \dots \\ -\Delta u_m + c_{m1}u_1 + \dots + c_{mm}u_m & = f_m \end{cases}$$

where

$$(2.8) \quad C = \frac{1}{2}(D + D^t) \quad \text{i.e.} \quad C = (c_{ij}), \quad c_{ij} = \frac{1}{2}(d_{ij} + d_{ji})$$

So to study the sign of the quadratic form Q we can also use the properties of the symmetric system.

Therefore we review briefly the spectral theory for this kind of symmetric systems, and use it to prove some results that we need for the possible nonsymmetric system (2.2).

Remark 2.1. If system (2.2) is cooperative, respectively fully coupled, so is the associate symmetric system (2.7).

2.1. Spectral theory for symmetric systems. Let Ω be a bounded domain in \mathbb{R}^N , $N \geq 2$, and consider for $m \geq 1$ the Hilbert spaces $\mathbf{L}^2 = \mathbf{L}^2(\Omega) = (L^2(\Omega))^m$, $\mathbf{H}_0^1 = \mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^m$, where if $f = (f_1, \dots, f_m)$, $g = (g_1, \dots, g_m)$ the scalar products are defined by

$$(2.9) \quad \begin{aligned} (f, g)_{\mathbf{L}^2} &= \sum_{i=1}^m (f_i, g_i)_{L^2(\Omega)} = \sum_{i=1}^m \int_{\Omega} f_i g_i dx \\ (f, g)_{\mathbf{H}_0^1} &= \sum_{i=1}^m (f_i, g_i)_{H_0^1(\Omega)} = \sum_{i=1}^m \int_{\Omega} \nabla f_i \cdot \nabla g_i dx \end{aligned}$$

Let $C = C(x) = (c_{ij}(x))_{i,j=1}^m$ a symmetric matrix whose elements are bounded functions:

$$(2.10) \quad c_{ij} \in L^\infty \quad , \quad c_{ij} = c_{ji} \quad \text{a.e. in } \Omega$$

and consider the bilinear forms

$$(2.11) \quad B(U, \Phi) = \int_{\Omega} [\nabla U \cdot \nabla \Phi + C(U, \Phi)] = \int_{\Omega} \left[\sum_{i=1}^m \nabla u_i \cdot \nabla \phi_i + \sum_{i,j=1}^m c_{ij} u_i \phi_j \right]$$

and, for $\Lambda > 0$

$$(2.12) \quad \begin{aligned} B^\Lambda(U, \Phi) &= \int_{\Omega} [\nabla U \cdot \nabla \Phi + (C + \Lambda I)(U, \Phi)] \\ &= \int_{\Omega} \left[\sum_{i=1}^m (\nabla u_i \cdot \nabla \phi_i + \Lambda u_i \phi_i) + \sum_{i,j=1}^m c_{ij} u_i \phi_j \right] \end{aligned}$$

Since $c_{ij} \in L^\infty$ and $c_{ij} = c_{ji}$, B and B^Λ are continuous symmetric bilinear forms, and, since $|\int_{\Omega} c_{ij} u_i \phi_j| \leq c \int_{\Omega} (u_i^2 + \phi_j^2)$, there exists $\Lambda \geq 0$ such that B^Λ is coercive in \mathbf{H}_0^1 , i.e it is an equivalent scalar product in \mathbf{H}_0^1 .

By the Riesz representation theorem, identifying $F \in \mathbf{L}^2$ with the linear functional $U \in \mathbf{H}_0^1 \mapsto (U, F)_{\mathbf{L}^2}$, for any $F \in \mathbf{L}^2$ there exists a unique $U =: T F \in \mathbf{H}_0^1$ such that $\|U\|_{\mathbf{H}_0^1} \leq c \|F\|_{\mathbf{L}^2}$ and $B^\Lambda(U, \Phi) = (F, U)_{\mathbf{L}^2}$ for any $\Phi \in \mathbf{H}_0^1$, i.e. U is the unique weak solution of the system

$$(2.13) \quad \begin{cases} -\Delta U + (C(x) + \Lambda I) U = F & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$$

i.e.

$$\begin{cases} -\Delta u_1 + (c_{11} + \Lambda)u_1 + \dots + c_{1m}u_m = f_1 & \text{in } \Omega \\ \dots & \dots \\ -\Delta u_m + c_{m1}u_1 + \dots + (c_{mm} + \Lambda)u_m = f_m & \text{in } \Omega \\ u_1 = \dots = u_m = 0 & \text{on } \partial\Omega \end{cases}$$

Moreover $T : F \mapsto U$, maps \mathbf{L}^2 into \mathbf{L}^2 and is *compact* because of the compact embedding of \mathbf{H}_0^1 in \mathbf{L}^2 , it is a *positive* operator, since $(TF, F)_{\mathbf{L}^2} = (U, F)_{\mathbf{L}^2} = B^\Lambda(U, U) > 0$ if $F \neq 0$ which implies $U \neq 0$ (recall that B^Λ is an equivalent scalar product in \mathbf{H}_0^1), and, since C is symmetric, it is also *selfadjoint*. Indeed if $TF = U$, $TG = V$, i.e. $B^\Lambda(U, \Phi) = (F, U)_{\mathbf{L}^2}$, $B^\Lambda(V, \Phi) = (G, V)_{\mathbf{L}^2}$ for any $\Phi \in \mathbf{H}_0^1$, then $(TF, G)_{\mathbf{L}^2} = (U, G)_{\mathbf{L}^2} = (G, U)_{\mathbf{L}^2} = B^\Lambda(V, U) = B^\Lambda(U, V) = (F, V)_{\mathbf{L}^2} = (F, TG)_{\mathbf{L}^2}$.

Thus, by the spectral theory of positive compact selfadjoint operators in Hilbert spaces there exist a nonincreasing sequence $\{\mu_j^\Lambda\}$ of eigenvalues with $\lim_{j \rightarrow \infty} \mu_j^\Lambda = 0$ and a corresponding sequence $\{W^j\} \subset \mathbf{H}_0^1$ of eigenvectors such that $G(W^j) = \mu_j^\Lambda W^j$. Putting $\lambda_j^\Lambda = \frac{1}{\mu_j^\Lambda}$ then W_j solves the systems $-\Delta W^j + (C + \Lambda I)W^j = \lambda_j^\Lambda W^j$, $W^j = 0$ on $\partial\Omega$. Translating, and denoting by λ_j the differences $\lambda_j = \lambda_j^\Lambda - \Lambda$, we conclude that there exist a sequence $\{\lambda_j\}$ of eigenvalues, with $-\infty < \lambda_1 \leq \lambda_2 \leq \dots$, $\lim_{j \rightarrow +\infty} \lambda_j = +\infty$, and a corresponding sequence of eigenfunctions $\{W^j\}$ that weakly solve the systems

$$(2.14) \quad \begin{cases} -\Delta W^j + CW^j = \lambda_j W^j & \text{in } \Omega \\ W^j = 0 & \text{on } \partial\Omega \end{cases}$$

i.e. if $W^j = (w_1, \dots, w_m)$

$$\begin{cases} -\Delta w_1 + c_{11}w_1 + \dots + c_{1m}w_m & = \lambda_j w_1 \\ \dots\dots\dots & \dots \\ -\Delta w_m + c_{m1}w_1 + \dots + c_{mm}w_m & = \lambda_j w_m \end{cases}$$

Moreover by (scalar) elliptic regularity theory applied iteratively to each equation, the eigenfunctions W^j belong at least to $C^1(\Omega; \mathbb{R}^m)$.

We now collect in the next proposition the variational formulation and some properties of eigenvalues and eigenfunctions.

In what follows if Ω' is a subdomain of Ω we denote by $\lambda_k(\Omega')$ the eigenvalues of the same system with Ω substituted by Ω' .

PROPOSITION 2.1. *Suppose that $C = (c_{ij})_{i,j=1}^m$ satisfies (2.10), and let $\{\lambda_j\}$, $\{W^j\}$ be the sequences of eigenvalues and eigenfunctions that satisfy (2.14).*

Define the Rayleigh quotient

$$(2.15) \quad R(V) = \frac{B(V, V)}{(V, V)_{\mathbf{L}^2}} \quad \text{for } V \in \mathbf{H}_0^1, V \neq 0$$

with $B(.,.)$ as in (2.11) Then the following properties hold, where \mathbf{V}_k denotes a k -dimensional subspace of \mathbf{H}_0^1 and the orthogonality conditions $V \perp W_k$ or $V \perp \mathbf{V}_k$ stand for the orthogonality in \mathbf{L}^2 .

$$i) \lambda_1 = \min_{V \in \mathbf{H}_0^1, V \neq 0} R(V) = \min_{V \in \mathbf{H}_0^1, (V, V)_{\mathbf{L}^2} = 1} B(V, V)$$

- ii) $\lambda_m = \min_{V \in \mathbf{H}_0^1, V \neq 0, V \perp W_1, \dots, V \perp W_{m-1}} R(V)$
 $= \min_{V \in \mathbf{H}_0^1, (V, V)_{\mathbf{L}^2} = 1, V \perp W_1, \dots, V \perp W_{m-1}} B(V, V)$ if $m \geq 2$
- iii) $\lambda_m = \min_{\mathbf{V}_m} \max_{V \in \mathbf{V}_m, V \neq 0} R(V)$
- iv) $\lambda_m = \max_{\mathbf{V}_{m-1}} \min_{V \perp \mathbf{V}_{m-1}, V \neq 0} R(V)$
- v) If $W \in \mathbf{H}_0^1$, $W \neq 0$, and $R(W) = \lambda_1$, then W is an eigenfunction corresponding to λ_1 .
- vi) $\lim_{\text{meas}(\Omega') \rightarrow 0} \lambda_1(\Omega') = +\infty$
- vii) If the system is cooperative in Ω and W is a first eigenfunction, then W^+ and W^- are eigenfunctions, if they do not vanish.
- viii) If the system is fully coupled in Ω , then the first eigenfunction does not change sign in Ω and the first eigenvalue is simple, i.e. up to scalar multiplication there is only one eigenfunction corresponding to the first eigenvalue.
- ix) Assume that the system is fully coupled in Ω , $C' = (c'_{ij})_{i,j=1}^m$ is another matrix that satisfies (2.10), and let $\{\lambda'_k\}$ be the sequence of eigenvalues of the corresponding system. If $c_{ij} \geq c'_{ij}$ for any $i, j = 1, \dots, m$ then $\lambda_1 \geq \lambda'_1$.

Proof. As before let us consider for $\Lambda \geq 0$ the bilinear form $B_\Lambda(V, V) = B(V, V) + \Lambda(V, V)_{\mathbf{L}^2}$, which is an equivalent scalar product in \mathbf{H}_0^1 , and define $R_\Lambda(V) = \frac{B_\Lambda(V, V)}{(V, V)_{\mathbf{L}^2}}$ for $V \in \mathbf{H}_0^1$, $V \neq 0$. Since $R_\Lambda(V) = R(V) + \Lambda$, once the properties are proved for $B_\Lambda(V)$ (which is an equivalent scalar product in \mathbf{H}_0^1) and its eigenvalues Λ_j^Λ , we recover the results stated by translation. Therefore for simplicity of notations we assume from the beginning that $\Lambda = 0$ i.e. that $B(., .)$ is an equivalent scalar product in \mathbf{H}_0^1 .

The proofs of i) , . . . , vi) do not depend on cooperativeness and they are the same as the standard proofs in the scalar case (see [9], [12]), so we only sketch them.

- i) The sequence $\{W_j\}$ is an orthonormal basis of \mathbf{L}^2 , and since $B(W_k, W_j) = \Lambda_k(W_k, W_j)_{\mathbf{L}^2}$ (in particular $= 0$ if $k \neq j$), the sequence $\{(\Lambda_j)^{-\frac{1}{2}} W_j\}$ is an orthonormal basis of \mathbf{H}_0^1 . It follows that if $U = \sum_{k=1}^\infty d_k W_k$ is the Fourier expansion of a function U in \mathbf{L}^2 , the series converges to U in \mathbf{H}_0^1 as well. If now $(U, U)_{\mathbf{L}^2} = \sum_{k=1}^\infty d_k^2 = 1$, then $B(U, U) = \sum_k \lambda_k d_k^2 \geq \lambda_1 \sum_k d_k^2 = \lambda_1$ and i) follows.
- ii) If $V \perp W_1, \dots, W_{m-1}$ and $(V, V)_{\mathbf{L}^2} = 1$, then $V = \sum_{k=m}^\infty d_k W_k$ and as before $B(V, V) \geq \lambda_m$ and since $B(W_m, W_m) = \lambda_m$ ii) follows.
- iii) If $\dim(\mathbf{V}_m) = m$ and $\{V_1, \dots, V_m\}$ is a basis of \mathbf{V}_m , there exists a linear combination $0 \neq V = \sum_{i=1}^m \alpha_i V_i$ which is orthogonal to W_1, \dots, W_{m-1} (m coefficients and $m - 1$ unknown), so that by ii) we obtain that $\max_{V \in \mathbf{V}_m, V \neq 0} R(V) \geq \lambda_m$. On the other hand if $\mathbf{V}_m = \text{span}(W_1, W_m)$ then $\max_{V \in \mathbf{V}_m, V \neq 0} R(V) \leq \lambda_m$, so that iii) follows.
- iv) The proof is similar. If $\{V_1, \dots, V_{m-1}\}$ is a basis of an $m - 1$ -dimensional subspace \mathbf{V}_{m-1} , there exists a linear combination $0 \neq W =$

$\sum_{i=1}^m \alpha_i W_i$ of the first m eigenfunctions which is orthogonal to \mathbf{V}_{m-1} , and $R(W, W) \leq \lambda_m$. So $\min_{V \perp \in \mathbf{V}_{m-1}, V \neq 0} R(V) \leq \lambda_m$, but taking $\mathbf{V}_m = \text{span}(W_1, W_{m-1})$ then $\min_{V \perp \in \mathbf{V}_{m-1}, V \neq 0} R(V) \geq \lambda_m$, so that iv) follows.

v) By normalizing we can suppose that $(W, W)_{\mathbf{L}^2} = 1$. Let $V \in \mathbf{H}_0^1$, $t > 0$. Then by i) $R(W + tV) = \frac{B(W+tV, W+tV)}{(W+tV)_{\mathbf{L}^2}} \geq \lambda_1$, i.e. $B(W, W) + t^2 B(V, V) + 2tB(W, V) \geq \lambda_1 [(W, W)_{\mathbf{L}^2} + t^2(V, V)_{\mathbf{L}^2} + 2t(W, V)_{\mathbf{L}^2}] = \lambda_1 + \lambda_1 t^2(V, V) + 2t\lambda_1(W, V)$. Since $B(W, W) = \lambda_1$, dividing by t and letting $t \rightarrow 0$ we obtain that $B(W, V) \geq \lambda_1(W, V)_{\mathbf{L}^2}$ and changing V with $-V$ we deduce that $B(W, V) = \lambda_1(W, V)_{\mathbf{L}^2}$ for any $V \in \mathbf{H}_0^1$, i.e. W is a first eigenfunction.

vi) If $V \in \mathbf{H}_0^1(\Omega')$ there exists $C \geq 0$ such that

$$B_{\Omega'}(V, V) \geq \int_{\Omega'} |\nabla V|^2 dx - C \int_{\Omega'} |V|^2 dx,$$

while by Poincaré's inequality

$$\int_{\Omega'} |V|^2 dx \leq C' |\Omega'|^{\frac{2}{N}} \int_{\Omega'} |\nabla V|^2 dx, \text{ so that}$$

$$R_{\Omega'}(V) = \frac{B_{\Omega'}(V, V)}{\int_{\Omega'} |V|^2 dx} \geq \frac{\int_{\Omega'} |\nabla V|^2 dx}{\int_{\Omega'} |V|^2 dx} - C \geq \frac{1}{C' |\Omega'|^{\frac{2}{N}}} - C \rightarrow +\infty \text{ if } |\Omega'| \rightarrow 0.$$

vii) Multiply (2.14) by $W^+ = (w_1^+, \dots, w_m^+)$ and integrate. If in the i -th equation, multiplied by w_i^+ we write $w_j = w_j^+ - w_j^-$ for $j \neq i$ and recall that by cooperativeness $-c_{1j} w_j^- w_1^+ \geq 0$, we deduce that $B(W^+, W^+) \leq \lambda_1(W^+, W^+)$, so that by v) W^+ is a first eigenfunction. The same applies to W^- .

viii) The conclusion follows from the strong maximum principle, which is valid under the fully coupling hypothesis. In fact if W^+ does not vanish, it is a first eigenfunction by vii), and by the strong maximum principle (Theorem 2.1) is strictly positive in Ω , i.e. $W > 0$ in Ω if it is positive somewhere.

If W^1, W^2 are two eigenfunctions corresponding to λ_1 , they do not change sign in Ω , so that they can not be orthogonal in \mathbf{L}^2 . This implies that the first eigenvalue is simple.

ix) Let $W^1 = (w_1, \dots, w_m)$ the first eigenfunction for the system (2.14). Since the system is fully coupled, W^1 does not change sign, and by normalizing it, we can assume that $(W^1, W^1)_{\mathbf{L}^2} = 1$.

Denoting by B' the bilinear form corresponding to the matrix C' , since $w_i w_j \geq 0$ and $c_{ij} \geq c'_{ij}$ we get that

$$(2.16) \quad \lambda_1 = B(W^1, W^1) = \int_{\Omega} \left[\sum_{i=1}^m |\nabla w_i|^2 + \sum_{i,j=1}^m c_{ij} w_i w_j \right] dx \geq \int_{\Omega} \left[\sum_{i=1}^m |\nabla w_i|^2 + \sum_{i,j=1}^m c'_{ij} w_i w_j \right] dx = B'(W^1, W^1) \geq \lambda_1'$$

□

2.2. Weak Maximum Principles for Cooperative systems. Let us turn back to the (possibly) nonsymmetric cooperative system (2.2):

$$\begin{cases} -\Delta U + D(x)U = F & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$$

where the matrix $D = (d_{ij})_{i,j=1}^m$ satisfies

$$(2.17) \quad d_{ij} \in L^\infty(\Omega) \quad , \quad d_{ij} \leq 0, \quad \text{whenever } i \neq j$$

In the sequel we shall indicate by $\lambda_j^{(s)} = \lambda_j^{(s)}(-\Delta + D; \Omega)$ the eigenvalues of the associated symmetric system:

$$(2.7) \quad \begin{cases} -\Delta U + C(x)U = F & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$$

where $C = \frac{1}{2}(D + D^t)$ i.e. $c_{ij} = \frac{1}{2}(d_{ij} + d_{ji})$.

Analogously the corresponding eigenfunctions will be indicated by $W_j^{(s)}$. We also denote the bilinear form associated with the symmetric system (2.7) by

$$B^s(U, \Phi) = \int_{\Omega} [\nabla U \cdot \nabla \Phi + C(U, \Phi)] = \int_{\Omega} \left[\sum_{i=1}^m \nabla u_i \cdot \nabla \phi_i + \sum_{i,j=1}^m c_{ij} u_i \phi_j \right]$$

As already remarked, the quadratic form (2.6) associated to the system (2.2) coincides with that associated to system (2.7), i.e.

$$Q(\Psi; \Omega) = \int_{\Omega} |\nabla \Psi|^2 + D(x)(\Psi, \Psi) = B^s(\Psi, \Psi)$$

if $\Psi \in H_0^1(\Omega; \mathbb{R}^m)$.

DEFINITION 2.2. We say that the maximum principle holds for the operator $-\Delta + D$ in an open set $\Omega' \subseteq \Omega$ if any $U \in \mathbf{H}^1(\Omega')$ such that

- $U \leq 0$ on $\partial\Omega'$ (i.e. $U^+ \in \mathbf{H}_0^1(\Omega')$)
- $-\Delta U + D(x)U \leq 0$ in Ω' (i.e. $\int \nabla U \cdot \nabla \Phi + D(x)(U, \Phi) \leq 0$ for any nonnegative $\Phi \in \mathbf{H}_0^1(\Omega')$)

satisfies $U \leq 0$ a.e. in Ω .

Let us denote by $\lambda_j^{(s)}(\Omega') > 0$ the sequence of the eigenvalues of the symmetric system in an open set $\Omega' \subseteq \Omega$.

THEOREM 2.2. *[Sufficient conditions for weak maximum principle]*
Under the hypothesis (2.17), if $\lambda_1^{(s)}(\Omega') > 0$ then the maximum principle holds for $-\Delta + D$ in $\Omega' \subseteq \Omega$.

Proof. By the variational characterization of eigenvalues

$\lambda_1^{(s)} = \min_{V \in \mathbf{H}^1(\Omega'), V \neq 0} R(V) > 0$, so that $Q(V) = B^s(V, V) > 0$ for any $V \neq 0$ in $\mathbf{H}_0^1(\Omega')$. Suppose that $U \leq 0$ on $\partial\Omega'$, $-\Delta U + D(x)U \leq 0$ in Ω' . Then, testing the equation with $U^+ = (u_1^+, \dots, u_m^+)$, writing in the

i -th equation $u_j = u_j^+ - u_j^-$ for $i \neq j$, and recalling that $-c_{ij}u_i^+u_j^- \geq 0$ if $i \neq j$, we obtain that $B^s(U^+, U^+) \leq 0$, which implies $U^+ \equiv 0$ in Ω' . \square

Almost immediate consequences of the previous theorem are the following "Classical" and "Small measure" forms of the weak maximum principle (see [5], [8], [20], [22]).

COROLLARY 2.1. i) *If (2.3) holds and D is a.e. nonnegative definite in Ω' then the maximum principle holds for $-\Delta + D$ in Ω' .*
 ii) *There exists $\delta > 0$, depending on D , such that for any subdomain $\Omega' \subseteq \Omega$ the maximum principle holds for $-\Delta + D$ in $\Omega' \subseteq \Omega$ provided $|\Omega'| \leq \delta$.*

Proof. i) If the matrix D is nonnegative definite then $B^s(\Psi, \Psi) \geq \int_{\Omega} |\nabla \Psi|^2$, which implies that the first symmetric eigenvalue is positive, so that by Theorem 2.2 we get i).

ii) It is a consequence of the Poincaré's inequality through vi) of Proposition 2.1. \square

Obviously the converse of Theorem 2.2 holds if $D = C$ is symmetric: if the maximum principle holds for $-\Delta + C$ in Ω' then $\lambda_1^{(s)}(\Omega') > 0$. In fact if $\lambda_1^{(s)}(\Omega') \leq 0$ since the system is cooperative (and symmetric) there exists a nontrivial nonnegative first eigenfunction $\Phi_1 \geq 0$, $\Phi \not\equiv 0$, and the maximum principle does not hold, since $-\Delta \Phi_1 + C \Phi_1 = \lambda_1 \Phi_1 \leq 0$ in Ω' , $\Phi_1 = 0$ on $\partial\Omega'$, while $\Phi_1 \geq 0$ and $\Phi_1 \neq 0$. However this is not true for general nonsymmetric systems. Roughly speaking the reason is that there is an equivalence between the validity of the maximum principle for the operator $-\Delta + D$ and the positivity of its principal eigenvalue $\tilde{\lambda}_1$, whose definition is given below, and the inequality $\tilde{\lambda}_1(\Omega') \geq \lambda_1^{(s)}(\Omega')$, which can be strict, holds.

More precisely we recall that the *principal eigenvalue* of the operator $-\Delta + D$ in an open set $\Omega' \subseteq \Omega$ is defined as

$$(2.18) \quad \begin{aligned} \tilde{\lambda}_1(\Omega') &= \sup \{ \lambda \in \mathbb{R} : \exists \Psi \in W_{loc}^{2,N}(\Omega'; \mathbb{R}^m) \text{ s.t.} \\ &\quad \Psi > 0 - \Delta \Psi + D(x)\Psi - \lambda \Psi \geq 0 \text{ in } \Omega' \} \end{aligned}$$

(see [5] and the references therein, and also [3] for the case of scalar equations).

We then have:

PROPOSITION 2.2. *Suppose that the system (2.2) is fully coupled in an open set $\Omega' \subseteq \Omega$. Then:*

$$(2.19) \quad \begin{aligned} &\text{i) } \textit{there exists a positive eigenfunction } \Psi_1 \in W_{loc}^{2,N}(\Omega'; \mathbb{R}^m) \text{ which satisfies} \\ &-\Delta \Psi_1 + D(x)\Psi_1 = \tilde{\lambda}_1(\Omega')\Psi_1 \text{ in } \Omega, \quad \Psi_1 > 0 \text{ in } \Omega', \quad \Psi_1 = 0 \text{ on } \partial\Omega' \end{aligned}$$

Moreover the principal eigenvalue is simple, i.e. any function that satisfy (2.19) must be a multiple of Ψ_1 .

- ii) the maximum principle holds for the operator $-\Delta + D$ in Ω' if and only if $\tilde{\lambda}_1(\Omega') > 0$
- iii) if there exists a positive function $\Psi \in W_{loc}^{2,N}(\Omega'; \mathbb{R}^m)$ such that $\Psi > 0$, $-\Delta\Psi + D(x)\Psi \geq 0$ in Ω' , then either $\tilde{\lambda}_1(\Omega') > 0$ or $\tilde{\lambda}_1(\Omega') = 0$ and $\Psi = c\Psi_1$ for some constant c .
- iv) $\tilde{\lambda}_1(\Omega') \geq \lambda_1^{(s)}(\Omega')$, with equality if and only if Ψ_1 is also the first eigenfunction of the symmetric operator $-\Delta + C$ in Ω' , $C = \frac{1}{2}(D + D^t)$. If this is the case the equality $C(x)\Psi_1 = D(x)\Psi_1$ holds and, if $m = 2$, this implies that $d_{12} = d_{21}$.

Proof. We refer to [5] for the proofs of i) – iii). For what concerns iv) we observe that evaluating the quadratic form $Q(\Psi; \Omega')$ on Ψ_1 and recalling that it coincides with the quadratic form associated with the symmetric operator $-\Delta + C$, $C = \frac{1}{2}(D + D^t)$ we obtain that

$\tilde{\lambda}_1(\Omega') = \frac{B^s(\Psi_1, \Psi_1)}{(\Psi_1, \Psi_1)_{L^2}} \geq \lambda_1^{(s)}(\Omega')$ with equality if and only if Ψ_1 is the first symmetric eigenfunction, by Proposition 2.1 v). If this is the case, since Ψ_1 satisfies the system (2.19) and the system (2.2), the equality $D(x)\Psi_1 = C(x)\Psi_1$ follows. Since Ψ_1 is positive, if $m = 2$ the equality $d_{12} = d_{21}$ follows. \square

3. Results on semilinear systems

Let Ω be a bounded domain in \mathbb{R}^N , $F = (f_1, \dots, f_m) : \overline{\Omega} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ a $C^{1,\alpha}$ function and consider the semilinear elliptic system for the unknown vector valued function $U = (u_1, \dots, u_m)$ in Ω :

$$(3.1) \quad \begin{cases} -\Delta U = F(x, U) & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$$

i.e.

$$\begin{cases} -\Delta u_1 = f_1(x, u_1, \dots, u_m) & \text{in } \Omega \\ \dots\dots\dots & \dots \\ -\Delta u_m = f_m(x, u_1, \dots, u_m) & \text{in } \Omega \\ u_1 = \dots = u_m = 0 & \text{on } \partial\Omega \end{cases}$$

The system (1.1) is a particular case of this system, with radial dependence on x .

DEFINITION 3.1. • We say that the system (3.1) is cooperative or weakly coupled in an open set $\Omega' \subseteq \Omega$ if

$$\frac{\partial f_i}{\partial u_j}(x, u_1, \dots, u_m) \geq 0 \quad \text{for every } (x, u_1, \dots, u_m) \in \Omega' \times \mathbb{R}^m$$

and every $i, j = 1, \dots, m$ with $i \neq j$.

- We say that the system (3.1) is fully coupled along a solution U in $\Omega' \subseteq \Omega$ if it is cooperative in Ω' and in addition $\forall I, J \subset \{1, \dots, m\}$ such that $I \neq \emptyset, J \neq \emptyset, I \cap J = \emptyset, I \cup J = \{1, \dots, m\}$ there exist $i_0 \in I, j_0 \in J$ such that

$$\text{meas} \left(\{x \in \Omega' : \frac{\partial f_{i_0}}{\partial u_{j_0}}(x, U(x)) > 0\} \right) > 0$$

Let us recall the following definition, which makes sense also in possibly unbounded domains.

DEFINITION 3.2. i) Let U be a $C^2(\Omega; \mathbb{R}^m)$ solution of (3.1). We say that U is linearized stable (or that has zero Morse index) if the quadratic form

$$(3.2) \quad \begin{aligned} Q_U(\Psi; \Omega) &= \int_{\Omega} [|\nabla \Psi|^2 - J_F(x, U)(\Psi, \Psi)] dx = \\ &= \int_{\Omega} \left[\sum_{i=1}^m |\nabla \psi_i|^2 - \sum_{i,j=1}^m \frac{\partial f_i}{\partial u_j}(x, U(x)) \psi_i \psi_j \right] dx \geq 0 \end{aligned}$$

for any $\Psi = (\psi_1, \dots, \psi_m) \in C_c^1(\Omega; \mathbb{R}^m)$ where $J_F(x, U)$ is the jacobian matrix of F computed at U .

- ii) U has (linearized) Morse index equal to the integer $\mu = \mu(U) \geq 1$ if μ is the maximal dimension of a subspace of $C_c^1(\Omega; \mathbb{R}^m)$ where the quadratic form is negative definite.
- iii) U has infinite (linearized) Morse index if for any integer k there is a k -dimensional subspace of $C_c^1(\Omega; \mathbb{R}^m)$ where the quadratic form is negative definite.

As observed in Section 2, the quadratic form Q_U associated to the linearized operator at a solution U , i.e. to the linear operator

$$(3.3) \quad L_U(V) = -\Delta V - J_F(x, U)V$$

coincides with the quadratic form corresponding to the selfadjoint operator

$$(3.4) \quad \tilde{L}_U(V) = -\Delta V - \frac{1}{2} (J_F(x, U) + J_F^t(x, U)) V$$

where J_F^t is the transpose of the matrix J_F .

Hence if λ_k and W^k denote the *symmetric* eigenvalues and eigenfunctions of L_U , i.e. W^k satisfy

$$\begin{cases} -\Delta W^k + C W^k = \lambda_k W^k & \text{in } \Omega \\ W^k = 0 & \text{on } \partial\Omega, \end{cases}$$

where $C = c_{ij}(x)$, $c_{ij}(x) = -\frac{1}{2} \left[\frac{\partial f_i}{\partial u_j}(x, U(x)) + \frac{\partial f_j}{\partial u_i}(x, U(x)) \right]$, as in the scalar case we can prove the following

PROPOSITION 3.1. *Let Ω be a bounded domain in \mathbb{R}^N . Then the Morse index of a solution U to (3.1) equals the number of negative symmetric eigenvalues of the linearized operator L_U .*

Proof. Let us denote by $\mu(U)$ the Morse index as previously defined, and by $m(U)$ the number of negative symmetric eigenvalues.

If Q_U is negative definite on a m -dimensional subspace of $C_c^1(\Omega; \mathbb{R}^m)$, by Proposition 2.1 iii) the m -th symmetric eigenvalue λ_m is negative, so that $m(U) \geq \mu(U)$.

On the other hand if there are m negative eigenvalues of the symmetric operator \tilde{L}_U in Ω , by the continuity of the eigenvalues there exists a subdomain $\Omega' \subset \Omega$ where there are m negative eigenvalues and corresponding orthogonal eigenfunctions W^1, \dots, W^m which by trivial extension can be considered as functions with compact support in Ω . Regularizing these functions we get that the quadratic form Q_U is negative definite on a subspace of $C_c^1(\Omega; \mathbb{R}^m)$ spanned by m linear independent functions, so that $\mu(U) \geq m(U)$. \square

3.1. Preliminary results. Let $e \in S^{N-1}$ be a direction, i.e. $e \in \mathbb{R}^N$, $|e| = 1$, and let us define the hyperplane $T(e)$ and the "cap" $\Omega(e)$ as

$$T(e) = \{x \in \mathbb{R}^N : x \cdot e = 0\}, \quad \Omega(e) = \{x \in \Omega : x \cdot e > 0\}$$

Moreover if $x \in \Omega$ let us denote by $\sigma_e(x)$ the reflection of x through the hyperplane $T(e)$ and by U^{σ_e} the function $U \circ \sigma_e$.

LEMMA 3.1. *Assume that U is any solution of (1.1) and that the hypotheses i)–iii) of Theorem 1.1 hold. Then, for any direction $e \in S^{N-1}$, the function $W^e = U - U^{\sigma_e} = (w_1, \dots, w_m)$ satisfies in $\Omega(e)$ (and in Ω) a linear system*

$$(3.5) \quad \begin{cases} -\Delta W + B_e(x)W &= 0 & \text{in } \Omega(e) \\ W &= 0 & \text{on } \partial\Omega(e) \end{cases}$$

i.e., if $B_e = (b_{ij})_{i,j=1}^m$,

$$\begin{cases} -\Delta w_1 + b_{11}w_1 + \dots + b_{1m}w_m = 0 & \text{in } \Omega(e) \\ \dots & \dots \\ -\Delta w_m + b_{m1}w_1 + \dots + b_{mm}w_m = 0 & \text{in } \Omega(e) \\ w_1 = \dots = w_m = 0 & \text{on } \partial\Omega(e), \end{cases}$$

such that for any $i, j = 1, \dots, m$ and $x \in \Omega$ the following hold:

$$(3.6) \quad b_{ii}(x) \geq -\frac{\partial f_i}{\partial u_i}(|x|, U(x)) \quad \text{if } u_i(x) \geq u_i^{\sigma_e}(x),$$

$$(3.7) \quad b_{ij}(x) \geq -\frac{\partial f_i}{\partial u_j}(|x|, U(x)) \quad \text{if } u_i(x) \geq u_i^{\sigma_e}(x), u_j(x) \geq u_j^{\sigma_e}(x)$$

while if $u_i(x) = u_i^{\sigma_e}(x)$, $u_j(x) = u_j^{\sigma_e}(x)$ then $b_{ij}(x) = -\frac{\partial f_i}{\partial u_j}(|x|, U(x))$.

Moreover the system (3.5) is fully coupled along W^e in $\Omega(e)$.

Proof. From the equation $-\Delta U = F(|x|, U(x))$ we deduce that the reflected function U_{σ_e} satisfies the equation $-\Delta U^{\sigma_e} = F(|x|, U^{\sigma_e}(x))$ and hence the difference $W^e = U - U^{\sigma_e}$ satisfies $-\Delta W^e = F(|x|, U) - F(|x|, U^{\sigma_e})$.

We can write the first equation of this system as

$$\begin{aligned} -\Delta w_1 &= [f_1(|x|, u_1, u_2, \dots, u_m) - f_1(|x|, u_1^{\sigma_e}, u_2, \dots, u_m)] + \\ &\quad [f_1(|x|, u_1^{\sigma_e}, u_2, \dots, u_m) - f_1(|x|, u_1^{\sigma_e}, u_2^{\sigma_e}, \dots, u_m)] + \\ &\quad \dots [f_1(|x|, u_1^{\sigma_e}, u_2^{\sigma_e}, \dots, u_{m-1}^{\sigma_e}, u_m) - f_1(|x|, u_1^{\sigma_e}, u_2^{\sigma_e}, \dots, u_m^{\sigma_e})] \end{aligned}$$

Then we define, using iii):

$$b_{11}(x) = - \int_0^1 \sum_{k \neq 1} \frac{\partial g_{1k}}{\partial u_1} [|x|, tu_1(x) + (1-t)u_1(x^{\sigma_e}), u_k(x)] dt ,$$

$$b_{12}(x) = - \int_0^1 \frac{\partial g_{12}}{\partial u_2} [|x|, u_1(x^{\sigma_e}), tu_2(x) + (1-t)u_2(x^{\sigma_e})] dt ,$$

...

$$b_{1m}(x) = - \int_0^1 \frac{\partial g_{1m}}{\partial u_m} [|x|, u_1(x^{\sigma_e}), tu_m(x) + (1-t)u_m(x^{\sigma_e})] dt$$

Hence we can write the equation as

$$-\Delta w_1 + b_{11}(x)(u_1 - u_1^{\sigma_e}) + \dots + b_{1m}(x)(u_m - u_m^{\sigma_e}) = 0$$

Proceeding analogously for the other equations and letting i playing the same role of 1 in the first equation, we get that W^e satisfies the linear system (3.5) with $B_e = (b_{ij})_{i,j=1}^m$ defined by

$$(3.8) \quad b_{ii}(x) = - \int_0^1 \sum_{k \neq i} \frac{\partial g_{ik}}{\partial u_i} [|x|, tu_i(x) + (1-t)u_i(x^{\sigma_e}), u_k(x)] dt$$

while if $j \neq i$

$$(3.9) \quad b_{ij}(x) = - \int_0^1 \frac{\partial g_{ij}}{\partial u_j} [|x|, u_i^{\sigma_e}, tu_j(x) + (1-t)u_j^{\sigma_e}(x)] dt$$

By ii) and iii), using Remark 1.2, we get (3.6), (3.7).

Obviously if $u_i(x) = u_i(x^{\sigma_e})$, $u_j(x) = u_j(x^{\sigma_e})$ then $b_{ij}(x) = -\frac{\partial f_i}{\partial u_j}(|x|, U(x))$.

Since the system (1.1) is weakly coupled in $\Omega(e)$, $\frac{\partial f_i}{\partial u_j} = \frac{\partial g_{ij}}{\partial u_j} \geq 0$ for any $i \neq j$, so that $b_{ij} \leq 0$ and the system (3.5) is weakly coupled in $\Omega(e)$ as well.

The system (3.5) is fully coupled in $\Omega(e)$ because if I, J are as in Definition 3.1, there exist $i_0 \in I$, $j_0 \in J$, $\bar{x} \in \Omega(-e)$, such that $\frac{\partial f_{i_0}}{\partial u_{j_0}}(|\bar{x}|, U(\bar{x})) = \frac{\partial g_{i_0 j_0}}{\partial u_{j_0}}(|\bar{x}|, u_{i_0}(\bar{x}), u_{j_0}(\bar{x})) > 0$, since by hypothesis i)

the system (1.1) is fully coupled along U in $\Omega(-e)$.

Then by the nonnegativity and continuity of $\frac{\partial g_{i_0 j_0}}{\partial u_{j_0}}$ also

$$b_{i_0 j_0}(\bar{x}^{\sigma_e}) = - \int_0^1 \frac{\partial g_{i_0 j_0}}{\partial u_{j_0}} [|\bar{x}|, u_{i_0}(\bar{x}), t u_{j_0}(\bar{x}^{\sigma_e}) + (1-t) u_{j_0}(\bar{x})] dt$$

is negative, with $\bar{x}^{\sigma_e} \in \Omega(e)$. \square

LEMMA 3.2. *Let $U = (u_1, \dots, u_m)$ be a solution of (1.1) and assume that the hypothesis i) of Theorem 1.1 holds. If for every $e \in S^{N-1}$ we have either $U \geq U^{\sigma_e}$ in $\Omega(e)$ or $U \leq U^{\sigma_e}$ in $\Omega(e)$, then U is foliated Schwarz symmetric.*

Proof. Let us start by proving that each component u_i of the solution is foliated Schwarz symmetric with respect to a vector $p_i \in \mathbb{R}^N$, $i = 1, \dots, m$. To do this we argue as in Lemma 2.4 in [11].

Let $p_i \in S^{N-1}$ be such that for some $r > 0$ $S_r = \{x \in \mathbb{R}^N : |x| = r\} \subset \Omega$ and $u_i(rp_i) = \max_{S_r} u_i$.

We define $T_{p_i}^+ = \{e \in S^{N-1} : e \cdot p_i > 0\}$ and $T_{p_i}^- = \{e \in S^{N-1} : e \cdot p_i < 0\}$. To prove that u_i is foliated Schwarz symmetric with respect to p_i it suffices to show that

$$(3.10) \quad u_i(x) \geq u_i(\sigma_e(x)) \quad \text{for all } x \in \Omega(e)$$

whenever $e \in T_{p_i}^+$.

Indeed this would imply the reverse inequality for $e \in T_{p_i}^-$, so that u_i would be axially symmetric about the axis with direction p_i , and the angular monotonicity would follow also by the same inequalities.

So for fixed $e \in T_{p_i}^+$ we consider the difference $w_i = u_i - u_i^{\sigma_e}$, and by assumption either $w_i \geq 0$, or $w_i \leq 0$. In the first case (3.10), is satisfied. In the second case, since the system (3.5) is cooperative, we have that $b_{ij} \leq 0$ if $i \neq j$ so that, since $w_j \leq 0$,

$$-\Delta w_i + b_{ii} w_i = - \sum_{i \neq j} b_{ij} w_j \leq 0, \quad w_i \leq 0 \quad \text{in } \Omega(e)$$

Then, by the (scalar) Strong Maximum Principle, we get that either $w_i < 0$ or $w_i \equiv 0$ in $\Omega(e)$, and the first case is not possible, since $rp_i \in \Omega(e)$ and $w_i(rp_i) \geq 0$ because $u_i(rp_i)$ is the maximum of u_i on S_r . Hence $w_i \equiv 0$ in $\Omega(e)$ and (3.10) is satisfied.

At this point, to prove that the solution U is foliated Schwarz symmetric we only need to prove that the vectors p_i are the same vector for all $i = 1, \dots, m$. To this aim consider the vector p_1 and take any hyperplane $T(e) = \{x \in \mathbb{R}^N : x \cdot e = 0\}$ passing through the axis with direction p_1 . The difference $W^e = U - U^{\sigma_e} = (w_1, \dots, w_m)$ satisfies in $\Omega(e)$ the linear system (3.5), which is fully coupled along W^e in $\Omega(e)$. Moreover either $U \geq U^{\sigma_e}$ in $\Omega(e)$ or $U \leq U^{\sigma_e}$ in $\Omega(e)$ by hypothesis, and $w_1 \equiv 0$ in $\Omega(e)$. So by Theorem 2.1 necessarily we have that $w_j \equiv 0$ in $\Omega(e)$ for all $j = 1, \dots, m$, i.e. all vectors p_j coincide with p_1 . \square

LEMMA 3.3. *Let $U = (u_1, \dots, u_m)$ be a solution of (1.1) and assume that the hypothesis i) of Theorem 1.1 holds.*

Suppose that there exists a direction e such that U is symmetric with respect to $T(e)$ and the principal eigenvalue $\tilde{\lambda}_1(\Omega(e))$ of the linearized operator $L_U(V) = -\Delta V - J_F(x, U)V$ in $\Omega(e)$ is nonnegative. Then U is foliated Schwarz symmetric.

Proof. We adapt to the case of systems the proof of Proposition 2.3 in [19].

After a rotation, we may assume that $e = e_2 = (0, 1, \dots, 0)$ so that $T(e) = \{x \in \mathbb{R}^N : x_2 = 0\}$. To prove the assertion we will use Lemma 3.2, so we consider a direction $e' \in S^{N-1}$, $e' \neq e$. After another orthogonal transformation which leaves e_2 and $T(e_2)$ invariant, we may assume that $e' = (\cos \theta_0, \sin \theta_0, 0, \dots, 0)$, for some $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Now we choose new coordinates, replacing (x_1, x_2) by polar coordinates (r, θ) , $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, and leaving $\tilde{x} = (x_3, \dots, x_N)$ unchanged.

The derivative U_θ of U with respect to θ , extended in the origin with $U_\theta(0) = 0$ if Ω is a ball, satisfies the linearized system, i.e.

$$(3.11) \quad -\Delta U_\theta - J_F(|x|, U)U_\theta = 0 \quad \text{in } \Omega(e_2)$$

Moreover, by the symmetry of U with respect to the hyperplane $T(e_2)$ we have that U_θ is antisymmetric with respect to $T(e_2)$ and therefore vanishes on $T(e_2)$ and since it vanishes on $\partial\Omega$, it vanishes on $\partial\Omega(e_2)$ as well.

Thus if $\tilde{\lambda}_1(\Omega(e)) > 0$, since the maximum principle holds, we get that $U_\theta \equiv 0$ in $\Omega(e)$.

If instead $\tilde{\lambda}_1(\Omega(e)) = 0$ and $U_\theta \not\equiv 0$, by the simplicity of the principal eigenvalue we get that U_θ is a principal eigenfunction and hence it is positive (or negative) in $\Omega(e)$.

In any case U_θ does not change sign in $\Omega(e)$, and arguing exactly as in Proposition 2.3 of [19] we get that for any direction e either $U \geq U^{\sigma_e}$ or $U \leq U^{\sigma_e}$. Thus, by Lemma 3.2, U is foliated Schwarz symmetric. \square

LEMMA 3.4. *Assume that U is a solution of (1.1) and that the hypotheses i)–iii) of Theorem 1.1 hold. Then*

$$Q_U((W^e)^+; \Omega(e)) = Q_U((U - U^{\sigma_e})^+; \Omega(e)) \leq 0 \quad \forall e \in S^{N-1}$$

where Q_U is the quadratic form defined in (3.2).

Proof. As we saw in Lemma 3.1 the difference $W^e = U - U^{\sigma_e}$ satisfies the linear system (3.5). Let us multiply the i -th equation of this system by $(u_i - u_i^{\sigma_e})^+$ and integrate. By Lemma 3.1 we get, since $b_{ik} \leq 0$ if $i \neq k$ and in the domain of integration where b_{ik} appears $u_i(x) \geq u_i^{\sigma_e}(x)$,

$u_k(x) \geq u_k^{\sigma_e}(x)$ (so that $b_{ik}(x) \geq -\frac{\partial f_i}{\partial u_k}(|x|, U(x))$) :

$$\begin{aligned}
 (3.12) \quad 0 &= \int_{\Omega} (|\nabla(u_i - u_i^{\sigma_e})^+|^2 + b_{ii}(x)[(u_i - u_i^{\sigma_e})^+]^2) + \\
 &\quad \sum_{k \neq i} \int_{\Omega} (b_{ik}(x)[(u_i - u_i^{\sigma_e})^+] [(u_k - u_k^{\sigma_e})^+ - (u_k - u_k^{\sigma_e})^-]) \geq \\
 &\quad \int_{\Omega} (|\nabla(u_i - u_i^{\sigma_e})^+|^2 + b_{ii}(x)[(u_i - u_i^{\sigma_e})^+]^2) + \\
 &\quad \sum_{k \neq i} \int_{\Omega} (b_{ik}(x)[(u_i - u_i^{\sigma_e})^+] [(u_k - u_k^{\sigma_e})^+]) \geq \\
 &\quad \int_{\Omega} \left(|\nabla(u_i - u_i^{\sigma_e})^+|^2 - \frac{\partial f_i}{\partial u_i}(|x|, u_1, u_2, \dots, u_m)[(u_i - u_i^{\sigma_e})^+]^2 \right) \\
 &\quad - \sum_{k \neq i} \int_{\Omega} \left(\frac{\partial f_i}{\partial u_k}(|x|, u_1, u_2, \dots, u_m)(u_i - u_i^{\sigma_e})^+(u_k - u_k^{\sigma_e})^+ \right)
 \end{aligned}$$

for any $i = 1, \dots, m$. Hence we obtain that

$$(3.13) \quad 0 \geq Q_U((u_1 - u_1^{\sigma_e})^+, \dots, (u_m - u_m^{\sigma_e})^+; \Omega(e))$$

□

LEMMA 3.5. *Suppose that U is a solution of (1.1) with Morse index $\mu(U) \leq N$ and assume that the hypothesis i) of Theorem 1.1 holds. Then there exists a direction $e \in S^{N-1}$ such that*

$$Q_U(\Psi; \Omega(e)) \geq 0$$

for any $\Psi \in C_c^1(\Omega(e); \mathbb{R}^m)$.

Equivalently the first symmetric eigenvalue $\lambda_1^s(L_u, \Omega(e))$ of the linearized operator $L_U(V) = -\Delta V - J_F(x, U)V$ in $\Omega(e)$ is nonnegative (and hence also the principal eigenvalue $\tilde{\lambda}_1(L_U, \Omega(e))$ is nonnegative).

Proof. The proof is immediate if $\mu(U) \leq 1$. In fact in this case for any direction e at least one amongst $\lambda_1^s(L_u, \Omega(e))$ and $\lambda_1^s(L_u, \Omega(-e))$ must be nonnegative, otherwise taking the corresponding first eigenfunctions we would obtain a 2-dimensional subspace of $C_c^1(\Omega; \mathbb{R}^m)$ where the quadratic form is negative definite.

So let us assume that $2 \leq j = \mu(U) \leq N$ and let Φ_1, \dots, Φ_j be mutually orthogonal eigenfunctions corresponding to the negative symmetric eigenvalues $\lambda_1^s(L_u, \Omega), \dots, \lambda_j^s(L_u, \Omega)$.

For any $e \in S^{N-1}$ let Φ_e be the first positive L^2 normalized eigenfunction of the symmetric system associated to the linearized operator $L_U(V)$ in $\Omega(e)$. We observe that Φ_e is uniquely determined by the

assumption i). Define

$$(3.14) \quad \Psi_e(x) = \begin{cases} \left(\frac{(\Phi_{-e}, \Phi_1)_{\mathbf{L}^2(\Omega(-e))}}{(\Phi_e, \Phi_1)_{\mathbf{L}^2(\Omega(e))}} \right)^{\frac{1}{2}} \Phi_e(x) & \text{if } x \in \Omega(e) \\ - \left(\frac{(\Phi_{-e}, \Phi_1)_{\mathbf{L}^2(\Omega(-e))}}{(\Phi_e, \Phi_1)_{\mathbf{L}^2(\Omega(e))}} \right)^{\frac{1}{2}} \Phi_{-e}(x) & \text{if } x \in \Omega(-e) \end{cases}$$

The mapping $e \mapsto \Psi_e$ is a continuous odd function from S^{N-1} to \mathbf{H}_0^1 and, by construction, $(\Psi_e, \Phi_1)_{\mathbf{L}^2(\Omega)} = 0$.

Therefore the mapping $h : S^{N-1} \rightarrow \mathbb{R}^{j-1}$ defined by

$$h(e) = ((\Psi_e, \Phi_2)_{\mathbf{L}^2(\Omega)}, \dots, (\Psi_e, \Phi_j)_{\mathbf{L}^2(\Omega)})$$

is an odd continuous mapping, and since $j - 1 < N$, by the Borsuk-Ulam Theorem it must have a zero. This means that there exists a direction $e \in S^{N-1}$ such that Ψ_e is orthogonal to all the eigenfunctions Φ_1, \dots, Φ_j . Since $\mu(U) = j$ this implies that $Q_U(\Psi_e; \Omega) \geq 0$, which in turn implies that either $Q_U(\Phi_e; \Omega(e)) \geq 0$ or $Q_U(\Phi_{-e}; \Omega(-e)) \geq 0$, i.e. either $\lambda_1^s(L_u, \Omega(e))$ or $\lambda_1^s(L_u, \Omega(-e))$ is nonnegative, so the assertion is proved. \square

3.2. Proof of the symmetry results.

Proof of Theorem 1.1. By Lemma 3.5 there exists a direction e such that the first symmetric eigenvalue $\lambda_1^s(L_u, \Omega(e))$ of the linearized operator is nonnegative, so that the principal eigenvalue $\tilde{\lambda}_1(\Omega(e))$ is nonnegative as well. Moreover by Lemma 3.4 we have that $Q_U((U - U^{\sigma_e})^+) \leq 0$, so that either $(U - U^{\sigma_e})^+ \equiv 0$, or $\lambda_1^s(L_u, \Omega(e)) = 0$ and $(U - U^{\sigma_e})^+$ is the positive first symmetric eigenfunction in $\Omega(e)$. In any case we have that either $U \leq U^{\sigma_e}$ or $U \geq U^{\sigma_e}$ in $\Omega(e)$ holds.

If $U \equiv U^{\sigma_e}$ in $\Omega(e)$ then by Lemma 3.3, since the principal eigenvalue $\tilde{\lambda}_1(\Omega(e))$ is nonnegative, we immediately obtain that the solution U is foliated Schwarz symmetric. If this is not the case and e.g. $U \geq U^{\sigma_e}$ then, by Lemma 3.1 we have that $U > U^{\sigma_e}$ in $\Omega(e)$.

We now apply, as in [19] and [11], the "rotating plane method", which is an adaptation of the Moving Plane method as developed in [2] and obtain a different direction e' such that U is symmetric with respect to $T(e')$ and the principal eigenvalue $\tilde{\lambda}_1(L_u, \Omega(e'))$ of the linearized operator in $\Omega(e')$ is nonnegative. Then by Lemma 3.3 we get that U is foliated Schwarz symmetric.

More precisely, without loss of generality we suppose that $e = (0, 0, \dots, 1)$ and for $\theta \geq 0$ we set $e_\theta = (\sin \theta, 0, \dots, \cos \theta)$, so that $e_0 = e$, and $\Omega_\theta = \Omega(e_\theta)$, $U^\theta = U^{\sigma_{e_\theta}}$, $W^\theta = U - U^{\sigma_{e_\theta}}$. Let us define $\theta_0 = \sup\{\theta \in [0, \pi) : U > U^\theta \text{ in } \Omega_\theta\}$. Then necessarily $\theta_0 < \pi$, since $(U - U^0)(x) = -(U - U^\pi)(\sigma_{e_\pi}(x))$ for any $x \in \Omega_0$ (and $\sigma_{e_\pi}(x) \in \Omega_\pi$).

Suppose by contradiction that $U \not\equiv U^{\theta_0}$ in Ω_{θ_0} . Then, by Lemma 3.1) applied to the difference $W^{\theta_0} = (U - U^{\theta_0})$, we get that $W^{\theta_0} > 0$ in Ω_{θ_0} . Taking a compact $K \subset \Omega_{\theta_0}$ with small measure and such that

(componentwise) $W^{\theta_0} > (\eta, \dots, \eta)$ for some $\eta > 0$, for θ close to θ_0 we still have that $W^\theta > (\frac{\eta}{2}, \dots, \frac{\eta}{2})$ in K , while $W^\theta > 0$ in $\Omega_\theta \setminus K$ by the weak maximum principle in domains with small measure. This implies that for θ greater than and close to θ_0 the inequality $U > U^\theta$ in Ω_θ still holds, contradicting the definition of θ_0 .

Therefore $U \equiv U^{\theta_0}$ in Ω_{θ_0} .

Observe that the difference W^θ satisfies the linear system (3.5) and does not change sign for any $\theta \in [0, \theta_0)$, which implies by Proposition 2.2 that it is the principal eigenfunction for the system (3.5) corresponding to the eigenvalue $\tilde{\lambda}_1 = 0$. As $\theta \rightarrow \theta_0$ the system (3.5) tends to the linearized system, in the sense that the coefficients $b_{ij}(x)$ tend to the derivatives $\frac{\partial f_i}{\partial u_j}(|x|, U(x))$. Then by continuity the principal eigenvalue $\tilde{\lambda}_1(\Omega(e_{\theta_0}))$ of the linearized operator $L_U(V) = -\Delta V - J_F(x, U)V$ in $\Omega(e_{\theta_0})$ is zero. \square

Proof of Theorem 1.2. Let us choose N orthogonal directions $e_1, e_2, \dots, e_N \in S^{N-1}$ and introduce new (cylinder) coordinates $(r, \theta, y_3, \dots, y_N)$ defined by the relations $x = r[\cos \theta e_1 + \sin \theta e_2] + \sum_{i=3}^N y_i e_i$.

Then the angular derivative U_θ of U with respect to θ , extended by zero at the origin if Ω is a ball, satisfies the linearized system, i.e.

$$(3.15) \quad \begin{cases} -\Delta U_\theta - J_F(|x|, U)U_\theta &= 0 & \text{in } \Omega \\ U_\theta &= 0 & \text{on } \partial\Omega \end{cases}$$

By the stability assumption we have that the first symmetric eigenvalue of the linearized system in Ω is nonnegative, so that by iv) of Proposition 2.2 also the principal eigenvalue $\lambda_1(\Omega)$ is nonnegative.

Then by (3.15) and Proposition 2.2 we get that U_θ , if does not vanish, must be the first symmetric eigenfunction corresponding to the eigenvalue 0 and hence it does not change sign in Ω . Since U_θ is 2π -periodic this is impossible and therefore $U_\theta \equiv 0$. By the arbitrariness of the vectors e_1, e_2 we conclude that U is radial. \square

Proof of Corollary 1.1. As remarked at the beginning of the proof of Lemma 3.5, if U is a Morse index one solution for any direction e either $\lambda_1^s(L_u, \Omega(e))$ or $\lambda_1^s(L_u, \Omega(-e))$ must be nonnegative. By the proof of Theorem 1.1 we can find a direction e such that U is symmetric with respect to the hyperplane $T(e)$ and the principal eigenvalue $\tilde{\lambda}_1(\Omega(e)) = \tilde{\lambda}_1(\Omega(-e)) \geq 0$. On the other hand by the symmetry $\lambda_1^s(\Omega(e)) = \lambda_1^s(\Omega(-e)) \geq 0$. If $\tilde{\lambda}_1(\Omega(e)) > 0$ then, by the maximum principle, the derivatives U_θ as defined in the proof of Lemma 3.3, must vanish, and hence U is radial. So if $U_\theta \not\equiv 0$ necessarily $\tilde{\lambda}_1(\Omega(e)) = \lambda_1^s(\Omega(e)) = 0$ and by v) of Proposition 2.2 U_θ is the first eigenfunction of the symmetrized system, as well as a solution of (3.15). So we get that $J_F(|x|, U)U_\theta = \frac{1}{2}(J_F(|x|, U) + J_F^t(|x|, U))U_\theta$, i.e. (1.3) and if $m = 2$, since U_θ is positive, we get (1.4). \square

4. Examples and comments

A first type of elliptic systems that could be considered are those of "gradient type" (see [7]), i.e. systems of the type (1.1) where $f_j(|x|, U) = \frac{\partial g}{\partial u_j}(|x|, U)$ for some scalar function $g \in C^{2,\alpha}([0, +\infty) \times \mathbb{R}^m)$.

In this case the solutions correspond to critical points of the functional

$$\Phi(u) = \frac{1}{2} \int_{\Omega} |\nabla U|^2 dx - \int_{\Omega} g(|x|, U) dx$$

in $\mathbf{H}_0^1(\Omega)$ and the linearized operator (3.3) coincides with the second derivative of Φ .

Thus standard variational methods apply which often give solutions of finite (linearized) Morse index, as, for example, in the case when the Mountain Pass Theorem can be used. So, if the hypotheses of Theorem 1.1 are satisfied, our symmetry results can be applied.

Many systems of this type have been studied (see [7] and the references therein). An example is the nonlinear Schrödinger system

$$(4.1) \quad \begin{cases} -\Delta u_1 + u_1 &= |u_1|^{2q-2}u_1 + b|u_2|^q|u_1|^{q-2}u_1 \text{ in } \Omega \\ -\Delta u_2 + \omega^2 u_2 &= |u_2|^{2q-2}u_2 + b|u_1|^q|u_2|^{q-2}u_1 \text{ in } \Omega \end{cases}$$

where $b > 0$, Ω is either a bounded domain or the whole \mathbb{R}^N , $N \geq 2$, $q > 1$ if $N = 2$ and $1 < q < \frac{N}{N-2}$ if $N \geq 3$. If $\Omega \neq \mathbb{R}^N$ the Dirichlet boundary conditions are imposed

$$(4.2) \quad u_1 = u_2 = 0 \text{ on } \partial\Omega$$

This type of system has been studied in several recent papers (see [1], [14] and the references therein).

It is easy to see that the system is of gradient type and the solutions of (4.1), (4.2) are critical points of the functional

$$(4.3) \quad I(U) = I(u_1, u_2) = \frac{1}{2} \int_{\Omega} |\nabla U|^2 dx - \frac{1}{2q} \int_{\Omega} (|u_1|^{2q} + |u_2|^{2q}) dx - \frac{b}{q} \int_{\Omega} |u_1 u_2|^q dx$$

in the space $\mathbf{H}_0^1(\Omega)$.

Thus the linearized operator (3.3) at a solution U corresponds to the second derivative of I in $\mathbf{H}_0^1(\Omega)$ and hence the corresponding linear system is symmetric.

Moreover, as observed in [14], the system (4.1) is cooperative, and fully coupled in $\Omega(e)$, for any $e \in S^{N-1}$, along every purely vector solution $U = (u_1, u_2)$, i.e. such that u_1 and u_2 are both not identically zero. In particular the system is fully coupled along positive solutions in any $\Omega(e)$.

Then Theorem 1.1 could be applied to any purely vector solution with Morse index less than or equal to N , in an annulus or in a ball. In particular we can consider solutions obtained by the Mountain Pass

Theorem which have Morse index not bigger than one which are purely vector positive solutions for suitable values of b and $q \geq 2$, as could be obtained by the same proof of [14] for the case $\Omega = \mathbb{R}^N$.

A second type of interesting systems of two equations are the so called "Hamiltonian type" systems (see [7] and the references therein). More precisely we consider the system

$$(4.4) \quad \begin{cases} -\Delta u_1 = f_1(|x|, u_1, u_2) & \text{in } \Omega \\ -\Delta u_2 = f_2(|x|, u_1, u_2) & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega \end{cases}$$

with

$$(4.5) \quad f_1(|x|, u_1, u_2) = \frac{\partial H}{\partial u_2}(|x|, u_1, u_2) \quad , \quad f_2(|x|, u_1, u_2) = \frac{\partial H}{\partial u_1}(|x|, u_1, u_2)$$

for some scalar function $H \in C^{2,\alpha}([0, +\infty) \times \mathbb{R}^2)$. These systems can be studied by considering the associated functional

$$(4.6) \quad J(U) = I(u_1, u_2) = \frac{1}{2} \int_{\Omega} \nabla u_1 \cdot \nabla u_2 \, dx - \int_{\Omega} H(|x|, u_1, u_2) \, dx$$

either in $\mathbf{H}_0^1(\Omega)$ or in other suitable Sobolev spaces (see [7]).

It is easy to see that the linearized operator defined in (3.3) does not correspond to the second derivative of the functional J , which is strongly indefinite. Nevertheless solutions of (4.4) can have finite linearized Morse index as we show with a few simple examples.

Let us consider the following system:

$$(4.7) \quad \begin{cases} -\Delta u_1 = \lambda e^{u_2} & \text{in } \Omega \\ -\Delta u_2 = \mu e^{u_1} & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega \end{cases}$$

where $\lambda, \mu \in \mathbb{R}$.

If $\lambda, \mu > 0$ then the system is fully coupled along every solution in any $\Omega(e)$, $e \in S^{N-1}$, and the hypotheses of Theorem 1.1 are satisfied.

Let us consider the function $G : \mathbb{R}^2 \times (C^{2,\alpha}(\Omega))^2 \rightarrow (C^{0,\alpha}(\Omega))^2$ defined by $G((\lambda, \mu), (u_1, u_2)) = (-\Delta u_1 - \lambda e^{u_2}, -\Delta u_2 - \mu e^{u_1})$.

We have that $G((0, 0), (0, 0)) = (0, 0)$, $\frac{\partial G}{\partial (u_1, u_2)}((0, 0), (0, 0))(\phi, \psi) = (-\Delta \phi, -\Delta \psi)$, and the first (symmetric) eigenvalue of this operator is strictly positive, so that the operator is invertible.

By the implicit function theorem for small λ, μ there is a unique non-trivial solution $(u_1(\lambda, \mu), u_2(\lambda, \mu))$ close to the trivial solution of the system (4.7) corresponding to $\lambda = \mu = 0$. Moreover the solution is positive by the maximum principle and it is linearized stable, so that it is radial. Indeed the first (symmetric) eigenvalue of the linearized operator corresponding to $\lambda = \mu = 0$, $u = v = 0$ is strictly positive, and by continuity it is positive for small λ, μ .

The same happens substituting the exponential with other nonlinearities $f(u_2)$, $g(u_1)$ with nonnegative derivative in a neighborhood of 0 and such that $f(0), g(0) > 0$.

We now consider, as another example, the system

$$(4.8) \quad \begin{cases} -\Delta u_1 &= u_2^p \text{ in } \Omega \\ -\Delta u_2 &= u_1^q \text{ in } \Omega \\ u_1, u_2 &> 0 \text{ in } \Omega \\ u_1 = u_2 &= 0 \text{ on } \partial\Omega \end{cases}$$

where $1 < p, q < \frac{N+2}{N-2}$. The idea is to proceed as in the previous example starting from the case $p = q$ and the solution $u_1 = u_2 = z$, where z is a scalar solution of the equation

$$(4.9) \quad \begin{cases} -\Delta z &= z^p \text{ in } \Omega \\ z &> 0 \text{ in } \Omega \\ z &= 0 \text{ on } \partial\Omega \end{cases}$$

which is nondegenerate.

Let us observe that if $p = q$ and z has Morse index equal to the integer $\mu(z)$, then $\mu(z)$ is also the Morse index of the solution $U = (u_1 = u_2) = (z, z)$ of the system (4.8). Indeed the linearized equation at z for the equation (4.9) and the linearized system at (z, z) for the system (4.8) are respectively

$$(4.10) \quad \begin{cases} -\Delta \phi - pz^{p-1}\phi &= 0 \text{ in } \Omega \\ \phi &= 0 \text{ on } \partial\Omega \end{cases}$$

and

$$(4.11) \quad \begin{cases} -\Delta \phi_1 - pz^{p-1}\phi_2 &= 0 \text{ in } \Omega \\ -\Delta \phi_2 - pz^{p-1}\phi_1 &= 0 \text{ in } \Omega \\ \phi_1 = \phi_2 &= 0 \text{ on } \partial\Omega \end{cases}$$

This implies that the eigenvalues of these two operators are the same, since if ϕ is an eigenfunction for (4.10) corresponding to the eigenvalue λ_k then taking $\phi_1 = \phi_2 = \phi$ we obtain an eigenfunction (ϕ_1, ϕ_2) for (4.11) corresponding to the same eigenvalue, while if (ϕ_1, ϕ_2) is an eigenfunction for (4.11) corresponding to the eigenvalue λ_k then $\phi = \phi_1 + \phi_2$ is an eigenfunction for (4.10) corresponding to the same eigenvalue.

So proceeding as in the previous example we can start from a nondegenerate solution of (4.9) with a fixed exponent $\bar{p} \in (1, \frac{N+2}{N-2})$ and find a branch of solutions of (4.8) corresponding to (possibly different) exponents p, q close to \bar{p} . For example if we start with a Mountain Pass (positive) solution z in the ball of equation (4.9) with the exponent \bar{p} ,

knowing that its Morse index is one and it is nondegenerate, we get a branch of Morse index one radial solutions for p, q close to \bar{p} .

If Ω is an annulus then a mountain pass solution z of (4.9) is foliated Schwarz symmetric, as shown in [17]. Then, if not radial, it is obviously degenerate. Nevertheless, working in spaces of symmetric functions, we could remove the degeneracy and apply the continuation method described above.

Thus, starting from an exponent \bar{p} for which the mountain pass solution z of (4.9) is not radial, we can construct solutions $U = (u_1, u_2)$ of (4.8) in correspondence of exponents p, q close to \bar{p} , with Morse index one. Then Theorem 1.1 applies and, in particular, we get that the coupling condition (1.4) holds, which in this case can be written as

$$(4.12) \quad pu_2^{p-1} = qu_1^{q-1} \quad \text{in } \Omega$$

Note that more generally, by Corollary 1.1, the equality (4.12) must hold for every solution $U = (u_1, u_2)$ of (4.8) with Morse index one giving so a sharp condition to be satisfied by the components of a solution of this type.

Let us remark that our results apply also when the nonlinearity depends on $|x|$ (in any way). Arguing as before it is not difficult to construct systems having solutions with low Morse index, in particular with Morse index one.

An example could be the "Henon system"

$$(4.13) \quad \begin{cases} -\Delta u_1 = |x|^\alpha u_2^p & \text{in } \Omega \\ -\Delta u_2 = |x|^\beta u_1^q & \text{in } \Omega \\ u_1, u_2 > 0 & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega \end{cases}$$

with $\alpha, \beta > 0, p, q > 1$.

Starting again from a solution of the scalar equation of mountain pass type it is possible to construct solutions of (4.13) with the same Morse index for p, q and α, β close to each other. Note that, for some values of α, β, p, q , such solutions would not be radial, even if Ω is a ball, but rather foliated Schwarz symmetric as for the scalar case.

Finally, as in the scalar case, we could consider nonhomogeneous equations and provide examples of convex nonlinearities for which there exist solutions of Morse index one which change sign and to which our results apply.

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